M smooth mfld, closed, dim $= n$.

g Riemannian metric (positive, symmetric bilinear form on each T_pM, smooth).

$\gamma : [0,1] \longrightarrow M$, $\quad \text{length}(\gamma) = \int_0^1 \sqrt{g(\dot\gamma, \dot\gamma)}$.

$u \in C^\infty(M)$, $\quad du = \dfrac{\partial u}{\partial x^i}\, dx^i$

$X \in \Gamma(TM)$, $\quad \nabla X = \left(\dfrac{\partial X^i}{\partial x^j} + \Gamma^i_{jk} X^k \right) \partial_i \otimes dx^j$

Levi-Civita connection

$\omega, \eta \in \Gamma(T^*M)$

$$(\nabla\omega)(X) = d\big(\omega(X)\big) - \omega(\nabla X)$$

$$\nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + \omega \otimes \nabla\eta.$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

$g^{il} g_{lj} = \delta^i_j$

Note we have $\nabla g = 0$. (parallel transport is an isometry of tangent spaces)

$$\nabla^2 u = \nabla du = \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u}{\partial x^k} \right) dx^i \otimes dx^j$$

$$\Delta u = tr(\nabla^2 u) = g^{ij} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u}{\partial x^k} \right).$$

On $\mathbb{R}^n$, $\quad \Delta u = \displaystyle\sum_{i=1}^n \frac{\partial^2 u}{\partial x^i \partial x^i}$

Curvature: $-R(X,Y,Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$

$$Ric(X,Z) = tr\, R(X,\cdot,Z,\cdot)$$

e_i ONB ← $= \sum_i R(X, e_i, Z, e_i)$

A smooth one-parameter family of metrics $g = g(t)$ solves Ricci flow if $\boxed{\frac{\partial g}{\partial t} = -2Ric}$

Many interesting applications e.g. Poincaré conjecture ($\pi_1(M^3)=0 \Rightarrow M^3 \cong S^3$).

Many open questions and potential future applications.

Exercise 1. In local coords x^i show that

$$-R_{ijkl} = -R(\partial_i,\partial_j,\partial_k,\partial_l) = g_{lm}\left(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}\right).$$

Conclude that

$$-2Ric_{ik} = -2g^{jl}R_{ijkl}$$
$$= g^{lm}\left(\partial_l\partial_m g_{ik} + \partial_i\partial_k g_{lm} - \partial_i\partial_m g_{lk} - \partial_l\partial_k g_{im}\right) + l.o.t.$$

[involving at most 1 derivative of g]

Exercise 2. Let $\Gamma_k = g_{kl}\,g^{ij}\,\Gamma^l_{ij}$. Show that

$$-(\partial_i\Gamma_k + \partial_k\Gamma_i) = g^{lm}\left(\partial_i\partial_k g_{lm} - \partial_i\partial_m g_{lk} - \partial_l\partial_k g_{im}\right) + l.o.t.$$

Conclude that if coordinates are harmonic i.e. $\Delta x^i = 0$ then
$-2 Ric_{ik} = \Delta g_{ik} + \text{l.o.t.}$

Thus, very informally, $\frac{\partial g}{\partial t} = -2 Ric$ is a kind of 'heat equation' for $g(t)$.

Recall: For (M, g) fixed, $u : M \times [0,T) \to \mathbb{R}$ solves heat equation if $\frac{\partial u}{\partial t} = \Delta u$.

Let $u(\cdot, 0) = u_0$. Choose ONB for $L^2(M)$ given by eigenfunctions $\Delta \varphi_m = -\lambda \varphi_m$.

Then $u_0 = \sum_{m=0}^{\infty} \langle u_0, \varphi_m \rangle \varphi_m$ and unique sol. to heat starting from u_0 is

$$u(x,t) = \sum_{m=0}^{\infty} e^{-\lambda_m t} \langle u_0, \varphi_m \rangle \varphi_m$$

$\therefore$ u decays exponentially to a const. function as $t \to \infty$!

Expectation: Heat-type equations rapidly 'uniformize' the solution.

Eells–Sampson 1964: Harmonic map heat flow for maps between Riem. mflds. $\longrightarrow$ This inspired Hamilton to introduce Ricci flow.

Unlike heat equation, Ricci flow is <u>nonlinear</u> and can form singularities in finite time.

Applications hinge on analysing these singularities to reveal the geometric and topological information they encode.

Examples.

1. If g_0 is Einstein i.e. $Ric(g_0) = \lambda g_0$ then $g(t) = (1-2\lambda t) g_0$ solves Ricci flow.

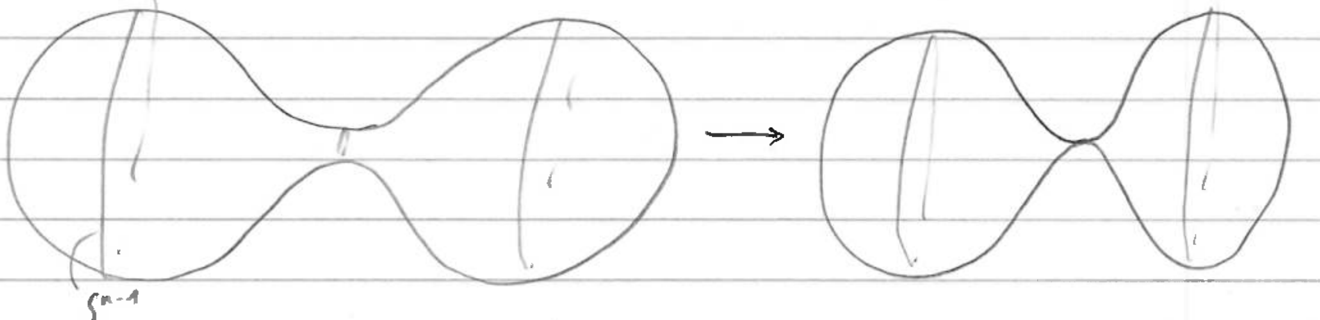
   $\lambda = 0 \Rightarrow g(t) = g_0$, $\lambda < 0 \Rightarrow g(t)$ 'expands',
   $\lambda > 0 \Rightarrow g(t)$ 'contracts'.

   Exercise 3 In case $\lambda > 0$ show that

   $$|Ric|^2_{g(t)} = g^{ik}(t)\, g^{jl}(t)\, Ric_{ij}(g(t))\, Ric_{kl}(g(t))$$

   blows up as $t \to 1/2\lambda$.

2. In every dimension $n \geq 3$ there exists an $O(n)$-invariant dumbbell such that



a 'neckpinch' sing. forms in finite time.

Short-time existence.

Theorem (Hamilton). Consider M closed, g_0 smooth. There is a $T>0$ and a smooth family of metrics $g(t)$, for $t\in[0,T)$, such that $g(t)$ solves Ricci flow and $g(0)=g_0$. Moreover, this solution is unique.

In local coords, $\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}$ is a 2^{nd}-order PDE, weakly parabolic, whereas we have a good existence theory for strongly parabolic equations.

De Turck trick: Solve a different eq. which is strongly parabolic and pull back by a flowing diffeo.

Fix a metric h on M. For $\varphi: M\to M$ define $\Delta_{g,h}\varphi = \mathrm{tr}_g(\nabla^2_{g,h}\varphi)$ where

$$\nabla^2_{g,h}\varphi(X,Y) := \nabla^h_{d\varphi(X)}(d\varphi(Y)) - d\varphi(\nabla^g_X Y).$$

Look for $\tilde{g}(t)$, $t\in[0,T)$, solving

$$\frac{\partial\tilde{g}}{\partial t} = -2\,\mathrm{Ric}(\tilde{g}) - \mathcal{L}_\xi\tilde{g} \quad \text{where}$$

'Ricci-De Turck flow'

$$\xi = \xi(t) := \Delta_{\tilde{g}(t),h}\, id.$$

Uniformly parabolic $\Rightarrow$ existence and uniqueness; $g(t) := \varphi_t^*\tilde{g}(t)$ is the desired Ricci flow for $\frac{d}{dt}\varphi_t = \xi_t$, $\varphi_0 = id$.

Note that computation showing Ricci–De Turck flow unif. parabolic is very similar to Exercise 2.

Evolution of curvature

If u solves heat equation then its derivatives solve heat-type equation with reaction terms dep. on curvature:

$$\frac{\partial}{\partial t} u = \Delta u \Rightarrow \frac{\partial}{\partial t} \nabla u = \nabla \frac{\partial u}{\partial t} = \nabla \Delta u = \Delta \nabla u + R * \nabla u$$

some contraction of R with ∇u.

Expectation: For $g(t)$ sol. to Ricci flow, curvature $R(g(t))$ solves heat-type eq. with reaction terms.

For $X = X(t)$ smooth time-dependent vector field on M, define

$$\frac{\partial X}{\partial t} = \frac{\partial X^i}{\partial t} \partial_i, \quad D_{\frac{\partial}{\partial t}} X = \frac{\partial X}{\partial t} - Ric(X, \partial_i) g^{ij} \partial_j.$$

Then $\left(D_{\frac{\partial}{\partial t}} g\right)(X,Y) = \frac{\partial}{\partial t} g(X,Y) - g\left(\frac{\partial X}{\partial t}, Y\right) - g\left(X, \frac{\partial Y}{\partial t}\right)$
$= 0.$

This simplifies many computations and formulae. We have:

$$\left(D_{\frac{\partial}{\partial t}} R\right)(X,Y,Z,W) = (\Delta R)(X,Y,Z,W) + Q(R)(X,Y,Z,W)$$

where $Q(R) = R * R$.

To get this formula one differentiates eq. for curvature, top order terms are of the form $\nabla^2 Ric$, convert these into ΔR using Bianchi.

Take trace:

$$\left(D_{\frac{\partial}{\partial t}} Ric\right)(X,Y) = (\Delta Ric)(X,Y) + 2\underbrace{R(X,\partial_i,Y,\partial_j)}_{}$$

$$g^{ik} g^{jl} Ric_{kl}$$

$$\frac{\partial}{\partial t}\, scal = \Delta\, scal + 2|Ric|^2.$$

<u>Smoothing estimates</u>

Differentiating $D_{\frac{\partial}{\partial t}} R = \Delta R + Q(R)$ one finds that $D_{\frac{\partial}{\partial t}} \nabla^m R = \Delta \nabla^m R + \sum_{l=0}^{m} \nabla^l R * \nabla^{m-l} R$.

Theorem (Shi). Suppose M compact, $g(t)$ sol. to Ricci flow for $t \in [0,1]$. Let $K := \sup_{t\in[0,1]} \sup_M |R(g(t))|_{g(t)}$. Then

$$\sup_M |\nabla^m R(g(t))|_{g(t)} \leq C\, t^{-m}$$

for all $t \in (0,1]$ and $m \geq 1$, where $C = C(n,m,K)$.

Note C does <u>not</u> depend on $\nabla^m R(g(0))$ in any way.

Exercise 4 For M compact, $g(t)$ smooth family of metrics (not necessarily Ricci flow), Suppose $u : M \times [t_1, t_2] \to \mathbb{R}$ is smooth and satisfies $\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u$. Show that

$$\max_M u(\cdot, t) \leq \max_M u(\cdot, t_1)$$

for all $t \in [t_1, t_2]$.

Exercise 5 Using evolution equations stated above, show that

$$\frac{\partial}{\partial t} |R|^2 \leq \Delta |R|^2 - 2|\nabla R|^2 + C_1(n) |R|^3 \quad \text{and}$$

$$\frac{\partial}{\partial t} |\nabla R|^2 \leq \Delta |\nabla R|^2 - 2|\nabla^2 R|^2 + C_2(n) |R| |\nabla R|^2.$$

Prove the Shi estimate in case $m = 1$ by applying Exercise 4 to the quantity

$$F = t^2 |\nabla R|^2 + a|R|^2 - bt$$

for suitable constants $a = a(n, K)$, $b = b(n, K)$.

Criterion for sing. formation

Theorem (Hamilton). M compact. Let $g(t)$, $t \in [0,T)$ be a maximal solution of Ricci flow. If $T < \infty$ then

$$\limsup_{t \to T} \sup_M |R(g(t))|_{g(t)} = \infty.$$

Idea: If false, use Shi estimates to extend smoothly to $t = T$, then continue flow using short-time existence theorem.

An application: 3-manifolds with $Ric > 0$

Theorem (Hamilton). $M = M^3$ compact. Suppose g_0 has positive Ricci curvature. Let $g(t)$, $t \in [0,T)$, be maximal Ricci flow with $g(0) = g_0$. Then $T < \infty$ and (up to renormalisation) $g(t)$ converges smoothly to a constant +ve curvature metric.

Corollary. $M \cong S^3/\Gamma$.

Exercise 6 When $n=3$ show that

$$\left(D_{\frac{\partial}{\partial t}} Ric\right)_{ij} = (\Delta Ric)_{ij} - 4 Ric_{ik} g^{kl} Ric_{lj} + 3\, scal\, Ric_{ij} + 2|Ric|^2 g_{ij} - scal^2 g_{ij}.$$

Exercise 7 Use the max. princ. and

$\frac{\partial}{\partial t} scal = \Delta scal + 2|Ric|^2$ to prove $scal > 0$ for all $t \in [0,T)$.

$\boxed{\text{Exercise 8}}$ Use Exercise 6 and the max. princ. to show that if the 'pinching' condition $Ric_{ij} \geq \varepsilon \cdot scal \cdot g_{ij}$ holds at $t=0$ (this must be true for some $\varepsilon > 0$), then it holds for all $t \in [0,T)$.

Hamilton then shows that pinching improves:

There are constants $\delta > 0$ and $C > 0$ depending only on g_0 such that

$$\frac{|Ric|^2 - \frac{1}{3} scal^2}{scal^2} \leq C \cdot scal^{-\delta}.$$

everywhere on M for all $t \in [0,T)$.

Remarks:

- Proven using Ex. 6 and max. princ.
- LHS is scale-invariant and vanishes iff $Ric_{ij} = \frac{1}{3} scal \cdot g_{ij}$
- Philosophically: curvature is 'becoming constant' at any point where $scal \longrightarrow \infty$.

This 'improvement of pinching' est. is decisive; further analysis yields the theorem.

Other applications in this vein: Brendle-Schoen proof of the differentiable sphere theorem.

See book, 'Ricci flow and the sphere theorem' by Brendle.